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SOME VARIATIONAL PRINCIPLES FOR LINEAR COUPLED THERMOELASTICITY

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Abstract—Guided by the principle of virtual work, the governing equations describing the physical behavior of a thermoelastic continuum were expressed as the Euler–Lagrange equations of certain variational principles. The differential variational principles were formulated for the thermoelastic continuum with or without an internal surface of discontinuity by introducing the dislocation potentials and Lagrange undetermined multipliers. These principles were shown to recover some of the earlier variational principles as special cases, and their reciprocals were also recorded.

1. INTRODUCTION

Universally, the equations governing the physical behavior of a continuum consist of the divergence (field) equations, the gradient equations, the constitutive relations, and the boundary and initial conditions. The field equations are originally stated in global (integral) form through the integral expressions of balance laws, and they can also be expressed in local (differential) form under certain regularity and differentiability conditions. The constitutive relations are given, excluding the nonlocal constitutive behavior, in differential form under certain rules and invariant requirements. The rest of the governing equations are almost always formulated in differential form. Alternatively, the governing equations given in integral and/or differential forms can be elegantly expressed in variational form, as the Euler-Lagrange equations of variational principles. The integral, differential and variational forms are, of course, equivalent and deducible from one another. An elaborate account of variational principles, including their existence, derivation and applications is reported by, among many others Lanczos (1960), Oden and Reddy (1976) and Washizu (1982).

Laying variational principles by an experienced guesswork aside, it was Biot (1956) who first derived a variational principle for the coupled problems of thermoelasticity starting with the basic thermodynamic laws of irreversible processes. Following Biot's principle, a large number of variational principles is developed and used, in particular, for deriving approximate direct solutions [see, for instance, the treatises of Boley and Weiner (1967), Biot (1970) and Nowacki (1986) and the survey papers by Carlson (1972) and Keramides (1983)]. Among those, Herrmann (1963) presented a generalized version of Biot's principle, as did Ben-Amoz (1965), in coupled thermoelasticity. By use of Gurtin's (1972) method of convolution, Iesan (1968), Nickell and Sackman (1968) and Rafalski (1968) formulated several variational principles. Besides, Batra (1989) proposed a principle of virtual work for thermoelastic bodies and associated quasi-variational principles. Recently, following a way developed by Biot (1956), Li (1992) derived a variational principle corresponding to the basic equations of thermoelasticity for an anisotropic medium. In view of the open literature cited, this paper addresses to a systematic derivation of certain differential variational principles for linear coupled thermoelasticity.

After introducing the notation used in the paper, a summary of the three dimensional differential equations is noted in the following section. The principle of virtual work is applied to an anisotropic thermoelastic region and an associated variational principle with certain constraint conditions is derived in Section 3. This two-field, differential variational principle is extended so as to incorporate the jump conditions in a thermoelastic region with an internal surface of discontinuity. By introducing the dislocation potentials and Lagrange undetermined multipliers, a unified variational principle is formulated in Section 4. The last section is devoted to concluding remarks and future needs of research.

Notation—yearning for its versatility and simplicity, the usual indicial notation is freely used in a three dimensional (3-D) Euclidean space Ξ. Einstein's summation convention is implied over all repeated Latin (1, 2, 3) and Greek (1, 2) indices, unless they are placed within parentheses. In the space Ξ, the x_i-system is identified with a fixed, right-handed system of Cartesian coordinates, and a comma stands for partial differentiation with respect to the indicated space coordinate and a superposed dot for time differentiation. An asterisk is used to indicate prescribed quantities and a bold face bracket to show the jump of an enclosed quantity across a surface of discontinuity. The symbol $\Omega(t)$ refers to a regular, finite and bounded region Ω of the space Ξ at time t, $\bar{\Omega}$ denotes the closure of the region Ω with its boundary surface $\partial\Omega$, and $\bar{\Omega}\times T$ represents the Cartesian product of the region Ω and the time interval $T = [t_0, t_1)$. Further, $C_{(mn)}$ refers to a class of continuous functions with its continuous derivatives of order up to and including (m) and (n) with respect to the space coordinates and time, respectively. Subscripts (m) and (h) are used to distinguish the denotations involving with the mechanical and thermal terms.

2. FUNDAMENTAL THREE-DIMENSIONAL DIFFERENTIAL EQUATIONS

A brief summary of the fundamental equations of thermoelasticity is given herein for ease of reference [e.g., Boley and Weiner (1967)]. Consider a regular, finite and bounded region Ω occupied by thermoelastic continuum in the space Ξ. The closure of region is denoted by $\bar{\Omega}$ and its entire boundary surface by $\partial\Omega$ which consists of the complementary regular subsurfaces $(\partial\Omega_u, \partial\Omega_t)$ or $(\partial\Omega_\theta, \partial\Omega_q)$. The term regular surface is used in the sense of Kellogg (1929). The region is referred to a fixed, right-handed system of Cartesian coordinates x_i. The domain of definitions for all the functions of the space coordinates x_i and time t is denoted by $\bar{\Omega}\times T$. Now, the differential equations of thermoelasticity are expressed by [e.g., Mason (1966)]:

Divergence equations (Cauchy's first and second laws of motion and the heat conduction equation)

$$L_j^{(m)} = t_{ij,i} + \rho(f_j - a_j) = 0 \quad \text{in } \bar{\Omega}\times T \tag{1a}$$

$$e_{ijk} t_{jk} = 0 \quad \text{in } \bar{\Omega}\times T \tag{1b}$$

and

$$L_{(h)} = q_{i,i} + \rho h + \Theta_o \dot{\eta} = 0 \quad \text{in } \bar{\Omega}\times T \tag{2}$$

with the definitions

- t_{ij} symmetric components of the stress tensor
- u_j components of the displacement vector
- a_j components of the acceleration vector $(\ddot{u}_j)$
- ρ mass density
- f_i components of the body force vector per unit mass
- e_{ijk} components of the alternating tensor
- q_i components of the heat flux vector
- h heat source per unit mass
- η entropy density
- Θ_o constant, positive, reference temperature; temperature of natural state in which stresses and strains do not exist.

Gradient equations (the strain–displacement relations and the heat strain–temperature relations)

$$L_{ij}^{(m)} = e_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in } \bar{\Omega} \times T$$

$$L_i^{(h)} = e_i - \theta_{,i} = 0 \quad \text{in } \bar{\Omega} \times T \tag{3}$$

with the definitions

e_{ij} components of the symmetric strain tensor
θ temperature increment from the reference temperature
e_i components of the heat strain vector.

Constitutive relations

$$K_{ij}^{(m)} = t_{ij} - \frac{\partial G}{\partial e_{ij}} = 0 \quad \text{in } \bar{\Omega} \times T \tag{4}$$

and

$$K_{(h)} = \eta + \frac{\partial G}{\partial \theta} = 0 \quad \text{in } \bar{\Omega} \times T \tag{5a}$$

$$K_i^{(h)} = q_i + \frac{\partial G}{\partial e_i} = 0 \quad \text{in } \bar{\Omega} \times T \tag{5b}$$

with the free energy function A, the dissipation function F and the thermoelastic function G of the form

$$G(e_{ij}, \theta_i, e_i) = A(e_{ij}, \theta_i) + F(e_i). \tag{6}$$

Looking to the linear constitutive relations, the quartic form of the functions is expressed by

$$A = \tfrac{1}{2}(c_{ijkl}e_{ij}e_{kl} - \rho C_v \Theta_o^{-1}\theta^2) - \lambda_{ij}e_{ij}\theta$$

$$F = \tfrac{1}{2}k_{ij}e_ie_j. \tag{7}$$

Here, c_{ijkl} stands for the second order elastic constants measured at constant field and temperature, λ_{ij} for the thermal stress constants relating an increase in temperature to a stress at constant strain or field, k_{ij} for the symmetric, positive semidefinite conductivity tensor, $\alpha = \rho C_v \Theta_o^{-1}$ for the linear thermal expansion coefficient and ρC_v for the specific heat per unit volume. Also, the usual symmetry relations for the material constants of the form

$$c_{ijkl} = c_{jikl} = c_{klij}, \lambda_{ij} = \lambda_{ji} \quad \text{in } \bar{\Omega} \times T \tag{8}$$

are written. In view of (4)–(7), the linear constitutive relations for the components of the stress tensor and the heat flux vector and for the entropy density in respective forms

$$M_{ij}^{(m)} = t_{ij} - (c_{ijkl}e_{kl} - \lambda_{ij}\theta) = 0 \quad \text{in } \bar{\Omega} \times T \tag{9}$$

and

$$M_{(h)} = \eta - (\lambda_{ij}e_{ij} + \alpha\theta) = 0 \quad \text{in } \bar{\Omega} \times T$$

$$M_i^{(h)} = q_i + k_{ij}e_j = 0 \quad \text{in } \bar{\Omega} \times T \tag{10}$$

are written. In (7)–(10), the elastic constants refer to the free constants since they describe the strain–stress relations when the thermal field is absent, while the remaining constants refer to the clamped constants [Venkataraman *et al.* (1975)].

Boundary conditions

$$L_i^{*(m)} = t_i - t_i^* = 0 \quad \text{on } \partial\Omega_t \times T \tag{11a}$$

$$L_{iu}^{*(m)} = u_i - u_i^* = 0 \quad \text{on } \partial\Omega_u \times T \tag{11b}$$

and

$$L_{(h)}^{*} = q - q_* = 0 \quad \text{on } \partial\Omega_q \times T \tag{12a}$$

$$L_{(h)\theta}^{*} = \theta - \theta_* = 0 \quad \text{on } \partial\Omega_\theta \times T \tag{12b}$$

together with the radiation condition of the form

$$L_{(h)r}^{*} = k\theta - q_* = 0 \quad \text{on } \partial\Omega_q \times T \tag{12c}$$

where k is a positive constant ranging from zero for an adiabatic boundary to infinity (i.e., $\theta = 0$) for an isothermal boundary, and

$t_j = n_i t_{ij}$ stress vector
$q = n_i q_i$ normal components of the heat flux vector across the boundary surface $\partial\Omega$.

Initial conditions

$$M_i^{*(m)} = u_i(x_j, t_0) - v_i^*(x_j) = 0 \quad \text{in } \Omega(t_0) \tag{13a}$$

$$N_i^{*(m)} = w_i^*(x_j) - \dot{u}_i(x_j, t_0) = 0 \quad \text{in } \Omega(t_0) \tag{13b}$$

and

$$M_{(h)}^{*} = \Theta_*(x_j) - \theta(x_j, t_0) = 0 \quad \text{in } \Omega(t_0). \tag{14}$$

Jump conditions

$$J_j^{(m)} = v_i[t_{ij}] = 0 \quad \text{on } S \times T \tag{15a}$$

$$J_i^{(um)} = [u_i] = 0 \quad \text{on } S \times T \tag{15b}$$

and

$$J_{(h)} = v_i[q_i] = 0 \quad \text{on } S \times T \tag{16a}$$

$$J_{(h\theta)} = [\theta] = 0 \quad \text{on } S \times T \tag{16b}$$

where S denotes a fixed, internal surface of discontinuity in the region Ω and v_i is the unit normal vector directed from the positive side of the discontinuity surface to its negative side, and the conventional notation for boldface brackets is introduced, namely,

$$[\chi] = \chi_i^+ - \chi_i^- = \chi_i^{(2)} - \chi_i^{(1)} \tag{17}$$

in which χ_i^+ and χ_i^- are the values of χ_i from the positive and negative sides of S.

Governing equations—the aforementioned eqns (1)–(16) completely describe the thermo-mechanical behavior of a nonlocal, nonpolar, elastic anisotropic continuum. The relativistic as well as quantum effects are excluded. The boundary and initial conditions (11)–(14) were shown to be sufficient for the existence and the uniqueness of solutions for the initial–mixed boundary value problems defined by the linear governing eqns (1)–(3) and (9)–(16).

3. THE PRINCIPLE OF VIRTUAL WORK AND AN ASSOCIATED DIFFERENTIAL VARIATIONAL PRINCIPLE

Among the general principles of mechanics, the principle of virtual work is chosen as the starting point in deriving variational principles of thermoelastic media. The principle of virtual work may be expressed as an assertion of the form

$$\delta L_g = -\delta\Sigma + \delta U + \delta^* W = 0 \tag{18a}$$

with

$$\delta\Sigma = \int_\Omega (t_{ij}\,\delta e_{ij} + q_i\,\delta e_i)\,\mathrm{d}V$$

$$\delta U = \int_\Omega [\rho(f_i - a_i)\,\delta u_i + (\Theta_0\dot{\eta} + \rho h)\,\delta\theta]\,\mathrm{d}V$$

$$\delta^* W = \int_{\partial\Omega} (t^i_*\,\delta u_i + q_*\,\delta\theta)\,\mathrm{d}S \tag{18b}$$

where $\delta^* W$ is the virtual work done by the external mechanical and thermal forces and an asterisk is placed upon this δ^* to distinguish it from the variation operator δ. Inserting (3) into (18b), carrying out the indicated variations, making use of the fact that the operation of variation commutes with that of differentiation, applying the divergence theorem for the regular region $\Omega + \partial\Omega$ and then rearranging terms in the surface and volume integrals, one arrives at the variational equation of the form

$$\delta L_g\{\Lambda_g = u_i, \theta\} = \int_\Omega (L_i^{(m)}\,\delta u_i + L_{(h)}\,\delta\theta)\,\mathrm{d}V - \int_{\partial\Omega} (L_i^{*(m)}\,\delta u_i + L^*_{(h)}\,\delta\theta)\,\mathrm{d}S = 0 \tag{19}$$

in which Cauchy's second law of motion [i.e., the symmetry of the stress tensor (1b)] is taken into account. Since (19) holds for any variations of the admissible state Λ_g, from setting all variations to zero, one obtains

$$L_i^{(m)} = L_h = 0 \quad \text{in } \bar{\Omega} \times T, \quad L_i^{*(m)} = L^*_{(h)} = 0 \quad \text{on } \partial\Omega \times T \tag{20}$$

in terms of the quantities defined by (1), (2), (11), and (12). This is a two-field differential variational principle which leads to Cauchy's first law of motion, the heat conduction equation and the associated natural boundary conditions of traction and heat flux. The rest of the fundamental equations of thermoelasticity remain as the constraint (subsidiary) conditions of the admissible state Λ_g. Next, (19) is integrated over the time interval $T = [t_0, t_1)$ with the result

$$\delta L_1\{\Lambda_g\} = \int_T \delta L_g\,\mathrm{d}t = \int_T [-\delta\Sigma + \delta K + \delta H + \delta^* W]\,\mathrm{d}t = 0 \tag{21}$$

which yields, as before, the two-field variational principle (30) under the condition

$$\delta u_i = 0, \quad \delta\theta = 0, \quad \text{in } \Omega(t_0) \text{ and } \Omega(t_1). \tag{22}$$

In this equation, the variations are assumed to obey the axiom of conservation of mass [i.e., $\delta(\rho\,\mathrm{d}v) = 0$] and the kinetic energy density $K\ (=\frac{1}{2}\rho\dot{u}_i\dot{u}_i)$ with its first variation for the region Ω as

$$\delta \int_T \mathrm{d}t \int_\Omega K \,\mathrm{d}v = -\int_T \mathrm{d}t \int_\Omega \rho \ddot{u}_i \,\delta u_i \,\mathrm{d}V = \int_T (\delta U - \delta H)\,\mathrm{d}t \tag{23}$$

is introduced under (22).

Yearning for a variational principle with as few constraint conditions as possible which is valuable from the standpoint of computational economy, the constraint conditions imposed upon the admissible state Λ_{ij} are now being removed. To begin with, consider a fixed, internal surface of discontinuity S in the region Ω and the associated jump conditions (15) and (16). To incorporate (15) and (16) into (21), following the procedure shown by Friedrichs (1929) [see also Courant and Hilbert (1953)], the dislocation potentials of the form

$$\Delta_{11}^{[1]} = \int_S \lambda_i J_i^{(um)}, \quad \Delta_{21}^{[2]} = \int_S \lambda J_{(h\theta)}\,\mathrm{d}S \tag{24}$$

are added to (18) as

$$\delta L_\lambda = \delta L_1 + \int_T \Delta_{\alpha 1}^{[\alpha]}\,\mathrm{d}t \tag{25}$$

where λ_i and λ are the Lagrangian undetermined multipliers. Performing the indicated variations in (25) and using the generalized version of the divergence theorem,

$$\int_\Omega \chi_{i,i}\,\mathrm{d}V = \int_{\partial\Omega - S_i} v_i \chi_i \,\mathrm{d}S - \int_S v_i [\chi_i]\,\mathrm{d}S \tag{26}$$

one finally arrives at the variational equation of the form [cf. Sarigül, Dökmeci (1984)],

$$\delta L_\lambda = \int_T \mathrm{d}t \sum_{\alpha=1}^{2} \left\{ \int_\Omega (L_i^{(m)}\,\delta u_i + L_{(h)}\,\delta\theta)\,\mathrm{d}V - \int_{\partial\Omega - S} (L_i^{*(m)}\,\delta u_i + L_{(h)}^{*}\,\delta\theta)\,\mathrm{d}S \right\}^{(\alpha)}$$
$$+ \int_T \mathrm{d}t \int_S \left\{ \delta\lambda_i [u_i] + \delta\lambda[\theta] + \sum_{\alpha=1}^{2} [(\lambda_i + v_j t_{ij}^{(\alpha)})\,\delta u_i^{(\alpha)} + (\lambda + v_i q_i^{(\alpha)})\,\delta\theta_{(\alpha)}](-1)^\alpha \right\} \mathrm{d}S = 0. \tag{27}$$

In (27), since the variations of $\Lambda_{ij}^{(\alpha)}$ are arbitrary, one reads (1) and (2) in $\bar{\Omega} \times T$ and (11a) and (12a) on $(\partial\Omega_\alpha - S)$, and since all the surface variations of λ_i and λ are now free, one has (15b) and (16b) and the conditions on $S \times T$, namely

$$(\lambda_i + v_j t_{ij}^{(\alpha)})(-1)^\alpha = 0, \quad (\lambda + v_i q_i^{(\alpha)})(-1)^\alpha = 0 \tag{28}$$

which gives the Lagrangian multipliers as

$$\lambda_j = -v_i \langle t_{ij} \rangle, \quad \lambda = -v_i \langle q_i \rangle \tag{29}$$

where the familiar symbolism $\langle\ \rangle$ indicates the mean value of enclosed quantity, namely,

$$\langle \chi_i \rangle = \tfrac{1}{2}\langle \chi_i^{(2)} + \chi_i^{(1)} \rangle \tag{30}$$

is introduced. Thus after substituting (29) into (27) and rearranging terms, one has the differential variational equation of the form

$$\delta L_{\gamma}\{\lambda_{\gamma}\} = \int_T \mathrm{d}t \sum_{\alpha=1}^{2} \left\{ \int_{\Omega} (L_i^{(m)}\,\delta u_i + L_{(h)}\,\delta\theta)\,\mathrm{d}V - \int_{\partial\Omega - S} (L_i^{*(m)}\,\delta u_i + L_{(h)}^{*}\,\delta\theta)\,\mathrm{d}S \right\}^{(\alpha)}$$

$$+ \int_T \mathrm{d}t \int_S v_i \{[t_{ij}]\langle\delta u_j\rangle - \langle\delta t_{ij}\rangle[u_j] + [q_i]\langle\delta\theta\rangle - \langle\delta q_i\rangle[\theta]\}\,\mathrm{d}S = 0 \quad (31a)$$

with

$$\Lambda_{\gamma} = \{u_i, \theta, t_{ij}, q_i\}^{(\alpha)} \quad (31b)$$

which leads to Cauch's first law of motion (1), the heat conduction equation (2), the boundary conditions (11a) and (12a), and the jump conditions (15) and (16), as its Euler–Lagrange equations, under the constraint conditions (1b), (3)–(5), (11b), (12b), (13), (14), and (22); conversely, if (1a), (2), (11a), (12a), (15) and (16) are met, the differential variational principle is evidently satisfied.

Although Friedrichs's transformation (or the Legendre transformation or the involutory transformation) [see e.g., Mura and Koya (1992)] is used in (41), a variety of methods is available in removing constraint (subsidiary) conditions in continuum physics. Noteworthy methods are due to Morse and Feshbach (1953) who advocated the adjoint equation method or the method of the mirror equation, Biot (1970) who introduced a quasi-variational method in Lagrangian thermodynamics and Gurtin (1972) who put forward the method of convolution in elastodynamics. Nevertheless, Friedrichs's transformation is widely used in mechanics, as is also implemented herein, owing to its versatility and rather easy applications [e.g., Chien (1984), Mang, Hoftetter and Gallagher (1985), and Tiersten (1969) who presented a lucid description of this transformation].

4. GENERALIZED VARIATIONAL PRINCIPLES

Starting with the principle of virtual work, the differential variational principle (31) with certain constraint conditions is derived in the previous section. These constraint conditions which make the choice of trial (approximating or coordinate) functions tedious, are now relaxed again by use of the involutory transformation. Accordingly, the Lagrange undetermined multipliers $(\mu_{ij}, \mu_i, \chi_i, \chi)^{(\alpha)}$ are introduced so as to incorporate (3)–(5), (11b), (12b), (13) and (14) and the dislocation potentials (24) and those given by

$$\Delta_{12}^{12(\alpha)} = \int_{\Omega} (\mu_{ij} L_{ij}^{(m)})^{(\alpha)}\,\mathrm{d}V$$

$$\Delta_{22}^{22(\alpha)} = \int_{\Omega} (\mu_i L_i^{(h)})^{(\alpha)}\,\mathrm{d}V \quad (32)$$

in the volume,

$$\Delta_{13}^{13(\alpha)} = \int_{\partial\Omega_{u(\alpha)}} (\chi_i L_{iu}^{*(m)})^{(\alpha)}\,\mathrm{d}S$$

$$\Delta_{23}^{23(\alpha)} = \int_{\partial\Omega_{\theta(\alpha)}} (\chi L_{(h)\theta}^{*})^{(\alpha)}\,\mathrm{d}S \quad (33)$$

on the surface are added to (18), namely,

$$\delta L_G\{\Lambda_G\} = \sum_{\alpha=1}^{2}\left\{\int_T \mathrm{d}t \int_\Omega \left(-\frac{\partial G}{\partial e_{ij}}\delta e_{ij} + \frac{\partial G}{\partial e_i}\delta e_i\right)\mathrm{d}V + \int_T \mathrm{d}t \int_\Omega \delta H \,\mathrm{d}V + \delta \int_T \mathrm{d}t \int_\Omega K\,\mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega} \delta^* W\,\mathrm{d}S\right\}^{(\alpha)} + \delta\int_T \mathrm{d}t \int_S (\lambda_i[u_i] + \lambda[\theta])\,\mathrm{d}S$$

$$+\delta\int_T \mathrm{d}t \sum_{\alpha=1}^{2}\left\{\int_\Omega (\mu_{ij}L_{ij}^{(m)} + \mu_i L_i^{(h)})\,\mathrm{d}V + \int_{\partial\Omega_u} \chi_i L_{iu}^{*(m)}\,\mathrm{d}S + \int_{\partial\Omega_\theta} \chi L_{(h)\theta}^{*}\,\mathrm{d}S\right\}^{(\alpha)} = 0 \quad (34)$$

in which (1b), (4) and (5) are considered and the multipliers represent a field of additional independent variables. As before, substituting (3) into this equation, carrying out the indicated variations and using the generalized Green–Gauss theorem, one finally obtains the variational equation of the form

$$\delta L_G\{\Lambda_G\} = \int_T \mathrm{d}t \sum_{\alpha=1}^{2}\left\{\int_\Omega\left[\left(\mu_{ij} - \frac{\partial G}{\partial e_{ij}}\right)\delta e_{ij} + \left(\mu_i + \frac{\partial G}{\partial e_i}\right)\delta e_i + L_j^{(m)}\,\delta u_j + L_{(h)}\,\delta\theta + \delta\mu_{ij}L_{ij}^{(m)} + \delta\mu_i L_i^{(h)}\right]\mathrm{d}V\right\}^{(\alpha)} + \int_T \mathrm{d}t \int_S (\delta\lambda_i J_i^{(um)} + \delta\lambda J_{(h\theta)} + \lambda_i\,\delta J_i^{(um)} + \lambda\,\delta J_{(h\theta)})\,\mathrm{d}S$$

$$+\int_T \mathrm{d}t \sum_{\alpha=1}^{2}\left\{\int_{\partial\Omega_u} (\delta\chi_i L_{iu}^{*(m)} + \chi_i\,\delta u_i)\,\mathrm{d}S + \int_{\partial\Omega_{(h)\theta}} (\delta\chi L_{(h)\theta}^{*} + \chi\,\delta\theta)\,\mathrm{d}S + \int_{\partial\Omega_t} L_j^{*(m)}\,\delta u_j\,\mathrm{d}S + \int_{\partial\Omega_{(h)}} L_{(h)}^{*}\,\delta\theta\,\mathrm{d}S\right\}^{(\alpha)} = 0 \quad (35)$$

in terms of the denotations (1)–(5), (11), (12), (15) and (16). In (35), the volumetric and surface variations of Lagrange multipliers are free in the region Ω, on the surface portions of $\partial\Omega$ and on the surface of discontinuity S, and hence, with the aid of (4) and (5), the Lagrange multipliers are identified by

$$\mu_{ij} = \frac{\partial G}{\partial e_{ij}} = t_{ij},\quad \mu_i = -\frac{\partial G}{\partial e_i} = q_i,\quad \chi_j = v_i t_{ij} = t_j,\quad \chi = v_i q_i = q$$
$$\lambda_j = -v_i\langle t_{ij}\rangle = -\langle t_j\rangle,\quad \lambda = -v_i\langle q_i\rangle = -\langle q\rangle. \quad (36)$$

Substitution of (36) into (35) gives the differential variational of the form

$$\delta L_G\,\{\Lambda_G\} = \sum_{\alpha=1}^{2}\int_T \mathrm{d}t(\delta L_G^{(m)} + \delta L_G^{(h)}) = 0 \quad (37a)$$

$$\Lambda_G = \{u_i, t_{ij}, e_{ij}, t_i\,; \theta, q_i, e_i, q\}^{(\alpha)} \quad (37b)$$

and

$$\delta L_{G(\alpha)}^{(m)} = \int_{\Omega(\alpha)}\left\{[t_{ij,i} + \rho(f_j - a_j)]\,\delta u_j + \left[e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right]\delta t_{ij} + \left(t_{ij} - \frac{\partial G}{\partial e_{ij}}\right)\delta e_{ij}\right\}^{(\alpha)}\mathrm{d}V$$
$$+\int_{\partial\Omega_{u(\alpha)}} [(u_i - u_i^*)\,\delta t_i]^{(\alpha)}\,\mathrm{d}S + \int_{\partial\Omega_{t(\alpha)}} [(v_i t_{ij} - t_j^*)\,\delta u_j]^{(\alpha)}\,\mathrm{d}S$$
$$+\int_S v_i\{[t_{ij}]\langle\delta u_j\rangle - \langle\delta t_{ij}\rangle[u_j]\}\,\mathrm{d}S \quad (38a)$$

and

$$\delta L^{(h)}_{G(\alpha)} = \int_{\Omega(\alpha)} \left\{ (q_{i,i} + \Theta_0 \dot{\eta} + \rho h)\, \delta\theta + (e_i - \theta,_i)\, \delta q_i + \left(q_i + \frac{\partial G}{\partial e_i} \right) \delta e_i \right\}^{(\alpha)} \mathrm{d}V$$

$$+ \int_{\partial\Omega^{(\alpha)}_{(h)\theta}} [(\theta - \theta^*)\, \delta q]^{(\alpha)}\, \mathrm{d}S + \int_{\partial\Omega^{(\alpha)}_{(h)}} [(v_i q_i - q^*)\, \delta\theta]^{(\alpha)}\, \mathrm{d}S + \int_S v_i \{ [q_i] \langle \delta\theta \rangle - \langle \delta q_i \rangle [\theta] \}\, \mathrm{d}S \quad (38\mathrm{b})$$

which generates (1a), (2), (3), (5b), (11), (12), (15) and (16) under the constraint conditions (1b), (5a), (13), (14) and (22), as its Euler–Lagrange equations. In (37), the term $L^*_{(h)}$ which represents the heat flux boundary conditions should be replaced by $L^*_{(h)_r}$ of (12c) for the radiation condition.

Keeping in mind the salient features of variational principles with as few constraints as possible, certain constraint conditions in (38) are removed below. To remove the constitutive constraint (5a), the dislocation potential of the form

$$\delta L^{11(\alpha)}_{11} = \int_T \mathrm{d}t \left\{ \int_\Omega K_{(h)}\, \delta\eta\, \mathrm{d}V \right\}^{(\alpha)} \quad (39)$$

is introduced. The initial conditions (15) and (16) are relaxed by the dislocation potentials of the form [see, Tiersten (1968) and Chen (1990)]

$$\delta L^{12(\alpha)}_{12} = \left\{ \int_\Omega N_i^{*(m)}\, \delta u_i(x_j, t_0)\, \mathrm{d}V \right\}^{(\alpha)}$$

$$\delta L^{21(\alpha)}_{21} = \left\{ \int_\Omega M_i^{*(m)}\, \delta \dot{u}_i(x_j, t_0)\, \mathrm{d}V \right\}^{(\alpha)} \quad (40\mathrm{a})$$

and

$$\delta L^{22(\alpha)}_{22} = \left\{ \int_\Omega M^*_{(h)}\, \delta\theta(x_j, t_0)\, \mathrm{d}V \right\}^{(\alpha)}. \quad (40\mathrm{b})$$

The dislocation potentials for (13) and (14) are obtained by simply using the Friedrichs's transformation in the time domain. This important point is discussed very thoroughly by Tiersten (1968) who was the first to find the dislocation potentials for the initial conditions in deriving a transformed version of Hamilton's principle, as did Chen (1990) who made the principle of total virtual action, the new foundation for all the time-integral variational statement/principles. Also, Simkins (1978, 1981) and Wu (1977, 1980) investigated a unified treatment of some initial value problems, including examples of application by use of finite elements. Now, by adding (39) and (40) to (37), one obtains a generalized variational principle under the constraint conditions (1b) and (22) as

$$\delta L_A \{\Lambda_A\} = \delta L_G + \sum_{\alpha=1}^{2} (\delta L^{\beta\sigma}_{\beta\sigma})^{(\alpha)} = 0 \quad (41\mathrm{a})$$

with

$$\Lambda_A = \{ \Lambda_G ; \eta, u_i(x_j, t_0), \dot{u}_i(x_j, t_0), \theta(x_j, t_0) \}^{(\alpha)}. \quad (41\mathrm{b})$$

Variational principle—a regular, finite and bounded thermoelastic region $\Omega + \partial\Omega + S$ with its piecewise smooth boundary surface $\partial\Omega_\alpha [= \partial\Omega_u \cup \partial\Omega_t, \partial\Omega_u \cap \partial\Omega_t = \phi ; = \partial\Omega_\theta \cup \partial\Omega_q, \partial\Omega_\theta \cap \partial\Omega_q = \phi]^{(\alpha)}$, its closure $\bar{\Omega}$ $(= \Omega \cup \partial B)$ and its internal surface of discontinuity S is considered in the space Ξ, and a differential variational principle (41) is given for this region. Then, of all the admissible states Λ_A which satisfy the symmetry of stress tensor (1b) and the condition (22) as well as the usual continuity, differentiability and existence conditions of field variables, if and only if, that admissible state Λ_A which satisfies the stress equations of motion (1a), the heat conduction equation (2), the strain–displacement relations (3a), the heat strain–temperature relations (3b), the constitutive relations (4), (5), the natural boundary conditions (11), (12), the natural initial conditions (13), (14) and the natural jump conditions (15), (16) is determined by the differential variational principle (41), $\delta L_A\{\Lambda_A\} = 0$, as it is a Euler–Lagrange equation. Conversely, if the aforementioned equations are met, the differential variational principle is evidently satisfied.

Recovered as special cases of (41), some of the earlier variational principles are now cited. The first special case is a fully linear, differential variational principle in which the constitutive terms $(K_{ij}^{(m)}, K^{(h)}, K_i^{(h)})$ of (4), (5) are replaced by their linear versions $(M_{ij}^{(m)}, M^{(h)}, M_i^{(h)})$ of (9), (10) in (41). Another noteworthy version is a variational principle of the form

$$\delta L_M\{\Lambda_M\} = \int_T \mathrm{d}t(\delta L_M^{(m)} + \delta L_M^{(h)}) = 0 \tag{42}$$

with the denotations of the form

$$\delta L_M^{(m)}\{\Lambda_M^{(m)}\} = \int_\Omega (L_i^{(m)}\,\delta u_i + L_{ij}^{(m)}\,\delta t_{ij} + K_{ij}^{(m)}\,\delta e_{ij})\,\mathrm{d}V + \int_{\partial\Omega_u} L_{iu}^{*(m)}\,\delta t_i\,\mathrm{d}S + \int_{\partial\Omega_t} L_i^{*(m)}\,\delta u_i\,\mathrm{d}S \tag{43a}$$

$$\delta L_M^{(h)}\{\Lambda_M^{(h)}\} = \int_\Omega (L_{(h)}\,\delta\theta + L_i^{(h)}\,\delta q_i + K_i^{(h)}\,\delta e_i + K_{(h)}\,\delta\eta)\,\mathrm{d}V + \int_{\partial\Omega_q} L_{(h)}^{*}\,\delta\theta\,\mathrm{d}S + \int_{\partial\Omega_\theta} L_{(h)\theta}^{*}\,\delta q\,\mathrm{d}S \tag{43b}$$

and

$$\Lambda_M = \Lambda_M^{(m)} \cup \Lambda_M^{(h)}; \quad \Lambda_M^{(m)} = \{u_i, t_{ij}, e_{ij}, t_i\}, \quad \Lambda_M^{(h)} = \{\theta, q_i, e_i, q, \eta\} \tag{44}$$

for the thermoelastic region $\Omega + \partial\Omega$ without a surface of discontinuity. The variational principle (42) is the counterpart of the Hu–Washizu principle [see e.g., Washizu (1982)] in coupled thermoelasticity. Similar variational principles are given by Herrmann (1963), Ben-Amoz (1965), Lardner (1963), Rafalski and Zyczkowski (1969), and Dökmeci (1978, 1980), and discussions of these by Lukasiewicz (1989). Besides, a number of variational principles analogous to those of linear elasticity can be readily derived from the generalized variational principle (41) [e.g., Dökmeci (1979, 1988) and Chien (1984)]. In the remaining part of this section, variational principles involving only certain field variables are given.

Denoting its admissible state $\Lambda_C\{t_i, q_i\}$, a variational principle involving only stresses and heat fluxes is recorded in the form

$$\delta L_C\{\Lambda_C\} = \int_T \mathrm{d}t \sum_{\alpha=1}^{2} \int_\Omega \{(L_{ij}^{(m)}\,\delta t_{ij} + L_i^{(h)}\,\delta q_i)\,\mathrm{d}V\}^{(\alpha)} = 0. \tag{45}$$

Another variational principle

$$\delta L_T\{\Lambda_T = u_i, \theta\} = \int_T \mathrm{d}t\left\{\int_\Omega (L_i^{(m)}\,\delta u_i + L_{(h)}\,\delta\theta)\,\mathrm{d}V + \int_{\partial\Omega_t} L_i^{*(m)}\,\delta u_i\,\mathrm{d}S + \int_{\partial\Omega_q} L_{(h)}^{*}\,\delta\theta\,\mathrm{d}S\right\} = 0 \tag{46}$$

is given, which operates on the mechanical displacements and the temperature field. Besides, the reciprocal of (31), with its admissible state $\Lambda_D = \{e_{ij}, t_i; e_i, q, \eta\}$, as

$$\delta L_D\{\Lambda_D\} = \int_T \mathrm{d}t \left\{ \int_\Omega (K_{ij}^{(m)}\,\delta e_{ij} + K_i^{(h)}\,\delta e_i + K_{(h)}\,\delta\eta)\,\mathrm{d}V + \int_{\partial\Omega_u} L_{iu}^{*(m)}\,\delta t_i\,\mathrm{d}S + \int_{\partial\Omega_\theta} L_{(h)\theta}^{*}\,\delta q\,\mathrm{d}S \right\} = 0 \quad (47)$$

is written.

5. CONCLUSION

Starting with the principle of virtual work and augmenting it through the dislocation potentials and Lagrange undetermined multipliers, the differential variational principles $\delta L_g\{\Lambda_g\} = 0$ in (19), $\delta L_l\{\Lambda_l\} = 0$ in (21), $\delta L_s\{\Lambda_s\} = 0$ in (27), $\delta L_G\{\Lambda_G\} = 0$ in (37) and $\delta L_A\{\Lambda_A\} = 0$ in (41) are derived. The latter, being the most general, is specialized so as to obtain the variational principle $\delta L_M\{\Lambda_M\} = 0$ in (42), $\delta L_C\{\Lambda_C\} = 0$ in (45), $\delta L_T\{\Lambda_T\} = 0$ in (46) and $\delta L_D\{\Lambda_D\} = 0$ in (47). The differential variational principle (41) generates all the governing equations of thermoelastic continua with a fixed, internal surface of discontinuity, including the natural initial and jump conditions, as its Euler–Lagrange equations. This variational principle is quite general, and has certain appealing features in succinctly summarizing the governing equations of continua, consistently deducing the lower order equations and, in particular, providing a standard basis for computation. Since the constraint conditions are relaxed, (41) allows one to make simultaneous approximation upon all the field variables. Thus, the burdensome choice of trial functions is relieved for complicated and/or time-dependent boundary conditions. The variational principles presented agree with and recover, as already indicated, some of earlier variational principles in elasticity and thermoelasticity in which the initial and jump conditions are, in general, left out of account.

Contemplating variational principles for polar, nonlocal, relativistic and alike continua [e.g., Dökmeci and Altay Askar (1994)] can be constructed, as in the derivation of the variational principle (41), since Friedrichs's transformation used herein possesses considerably broader applicability to holonomic as well as nonholonomic conditions [e.g., Lanczos (1964)]. Besides, the derivation can be extended for variational principles of thermoelastic continua having more than one internal surface of discontinuity and/or a moving internal surface of discontinuity. Also, the incremental motions as well as the geometrical non-linearities can be similarly treated.

Salient extensions cited above and the development of approximate direct solutions in conjunction with the variational principle (41) are the topics of continuing studies, and will be reported later.

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